

Stability of closed characteristics on compact convex hypersurfaces in \mathbf{R}^{2n}

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Abstract

Let $\Sigma \subset \mathbf{R}^{2n}$ with $n \geq 2$ be any C^2 compact convex hypersurface and only has finitely geometrically distinct closed characteristics. Based on Y.Long and C.Zhu's index jump methods [LZh], we prove that there are at least two geometrically distinct elliptic closed characteristics, and moreover, there exist at least $\varrho_n(\Sigma)$ ($\varrho_n(\Sigma) \geq [\frac{n}{2}] + 1$) geometrically distinct closed characteristics such that for any two elements among them, the ratio of their mean indices is irrational number.

Key words: Compact convex hypersurfaces, closed characteristics, stability, Maslov-type index

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Running head: Stability of closed characteristics

1 Introduction and main results

In this paper, let Σ be a fixed C^2 compact convex hypersurface in \mathbf{R}^{2n} , i.e., Σ is the boundary of a compact and strictly convex region U in \mathbf{R}^{2n} . We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose U contains the origin. We consider closed characteristics (τ, x) on Σ , which are solutions of the following problem

$$\begin{cases} \dot{x}(t) &= JN_{\Sigma}(x(t)), \quad x(t) \in \Sigma, \quad \forall t \in \mathbf{R}, \\ x(\tau) &= x(0), \end{cases} \quad (1.1)$$

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where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix in \mathbf{R}^n , $\tau > 0$, $N_\Sigma(x)$ is the outward normal vector of Σ at x normalized by the condition $N_\Sigma(x) \cdot x = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbf{R}^{2n}$. A closed characteristic (τ, x) is *prime*, if τ is the minimal period of x . Two closed characteristics (τ, x) and (σ, y) are *geometrically distinct*, if $x(\mathbf{R}) \neq y(\mathbf{R})$. We denote by $\mathcal{T}(\Sigma)$, the set of all geometrically distinct closed characteristics on Σ , and $[(\tau, x)]$ the set of all closed characteristics which are geometrically the same as (τ, x) . $\#A$ denotes the total number of elements in a set A .

Let $j : \mathbf{R}^{2n} \rightarrow \mathbf{R}$ be the gauge function of Σ , i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, then $j \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ and $\Sigma = j^{-1}(1)$. Fix a constant $\alpha \in (1, 2)$ and define the Hamiltonian function $H_\alpha : \mathbf{R}^{2n} \rightarrow [0, +\infty]$ by

$$H_\alpha(x) = j(x)^\alpha, \quad \forall x \in \mathbf{R}^{2n}. \quad (1.2)$$

Then $H_\alpha \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system

$$\begin{cases} \dot{x}(t) &= JH'_\alpha(x(t)), \quad H_\alpha(x(t)) = 1, \quad \forall t \in \mathbf{R}, \\ x(\tau) &= x(0). \end{cases} \quad (1.3)$$

Denote by $\mathcal{J}(\Sigma, \alpha)$ the set of all geometrically distinct solutions (τ, x) of (1.3) where τ is the minimal period of x . Note that elements in $\mathcal{T}(\Sigma)$ and $\mathcal{J}(\Sigma, \alpha)$ are one to one correspondent to each other.

Let $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, the fundamental solution $\gamma_x : [0, \tau] \rightarrow \text{Sp}(2n)$ with $\gamma_x(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{y}(t) = JH''_\alpha(x(t))y(t), \quad \text{for all } t \in \mathbf{R} \quad (1.4)$$

is called the associated symplectic path of (τ, x) . The eigenvalues of $\gamma_x(\tau)$ are called Floquet multipliers of (τ, x) . A closed characteristic (τ, x) is *non-degenerate*, if 1 is a Floquet multiplier of y of precisely algebraic multiplicity 2, and is *elliptic*, if all the Floquet multipliers of x are on $\mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$, i.e., the unit circle in the complex plane.

The study on closed characteristics on the star-shaped hypersurface in the global sense started in 1978 by Rabinowitz in [Rab] and Weinstein for the convex hypersurface independently [Wei]. For more results on the multiplicity of geometrically distinct closed characteristics on convex hypersurfaces, please refer to [EkL], [EkH], [Szu], [HWZ], [LZh], [LLZ], [WHL], etc. and references therein.

For the stability problem, Ekeland proved in [Eke2] the existence of at least one elliptic closed characteristic on Σ provided $\Sigma \in \mathcal{H}(2n)$ is $\sqrt{2}$ -pinched. In [DDE] of 1992, Dell'Antonio, D'Onofrio and Ekeland proved the existence of at least one elliptic closed characteristic on Σ provided $\Sigma \in \mathcal{H}(2n)$ satisfies $\Sigma = -\Sigma$. In [Lon4] of 2000, Long proved that $\Sigma \in \mathcal{H}(4)$ and $\#\mathcal{T}(\Sigma) = 2$ imply that both of the closed characteristics must be elliptic. In [LZh] of 2002, Long and Zhu further proved when $\#\mathcal{T}(\Sigma) < +\infty$, there exists at least one elliptic closed characteristic and there are at least $\lfloor \frac{n}{2} \rfloor$ geometrically distinct closed characteristics on Σ possessing irrational mean indices, which are then non-hyperbolic. Moreover, they proved there exist at least two elliptic closed characteristics provided that $\#\mathcal{T}(\Sigma) \leq 2\varrho_n(\Sigma) - 2 < \infty$, where $\varrho_n(\Sigma)$ is defined by Definition 2.10. In the recent paper [LWa], Long and Wang proved that there exist at least two non-hyperbolic closed characteristic on $\Sigma \in \mathcal{H}(6)$ when $\#\mathcal{T}(\Sigma) < +\infty$ and in [Wa], Wang proved that there exist at least two elliptic closed characteristic on $\Sigma \in \mathcal{H}(6)$ when $\#\mathcal{T}(\Sigma) = 3$. Other results please refer [Lon2], [Lon4], [LLW]. Motivated by these results, we prove the following results in this paper:

Theorem 1.1. For any $\Sigma \in \mathcal{H}(2n)$ with $n \geq 2$ satisfying $\#\mathcal{T}(\Sigma) < +\infty$, there exist at least two elliptic closed characteristics on Σ .

A typical example is the non-resonant ellipsoid in \mathbf{R}^{2n} , that is Σ is defined by

$$\sum_{i=1}^n \frac{\alpha_i}{2} (p_i^2 + q_i^2) = 1, \quad (1.5)$$

where $\alpha_i/\alpha_j \in \mathbf{R} \setminus \mathbf{Q}$. There just exist n closed characteristics $x_i, i = 1, \dots, n$ and their mean Maslov-type index satisfy $\hat{i}(x_i)/\hat{i}(x_j) = \alpha_j/\alpha_i \in \mathbf{R} \setminus \mathbf{Q}$. When $\#\mathcal{T}(\Sigma)$ is finite, it seems that all the Maslov-type index of the closed characteristics are similar to those in the non-resonant ellipsoid. Another example is in the case $n = 2$, it has been proved in [HWZ] that there are either infinite or 2 closed characteristics. When $n = 2$, in the case $\#\mathcal{T}(\Sigma) = 2$, Long, Wang and Hu have proved that both of their mean index are irrational number and all the iterations of their Maslov-type index are the same as a non-resonant ellipsoid, another different proof is given by [BCE]. As results toward this aspect, we proved that

Theorem 1.2. For any $\Sigma \in \mathcal{H}(2n)$ satisfying $\#\mathcal{T}(\Sigma) < +\infty$, there exist at least $\varrho_n(\Sigma)$ geometrically distinct closed characteristics on Σ such that any two element $[(\tau, x)], [(\tilde{\tau}, \tilde{x})]$ satisfy

$$\frac{\hat{i}(x, 1)}{\hat{i}(\tilde{x}, 1)} \in \mathbf{R} \setminus \mathbf{Q}. \quad (1.6)$$

The main ingredient in our proof of this theorems is the Maslov-type index iteration theory developed by Long and his coworkers, especially based on some new observations on the common index jump theorem of Long and Zhu (Theorem 4.3 of [LZh], cf. Theorem 11.2.1 of [Lon5]). In Section 2, we review briefly the variational structure and the common index jump theorem of Long and Zhu with some further discussion, we prove our main theorems in Section 3. For reader's convenience, we brief review the Maslov-type index theory in Section 4.

In this paper, let \mathbf{N} , \mathbf{Z} , \mathbf{Q} , \mathbf{Q}^+ , \mathbf{R} , and \mathbf{R}^+ denote the sets of natural integers, integers, rational numbers, positive rational number, real numbers, and positive real numbers respectively. Denote by (a, b) and $|a|$ the standard inner product and norm in \mathbf{R}^{2n} . Denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the standard L^2 inner product and L^2 norm. We also define the functions

$$\begin{cases} [x] &= \max\{k \in \mathbf{Z} | k \leq x\}, & E(x) = \min\{k \in \mathbf{Z} | k \geq x\}, \\ \{x\} &= x - [x], & \varphi(x) = E(x) - [x]. \end{cases} \quad (1.7)$$

2 Brief review of Long-Zhu index jump Theorem and with further discussion

To solve the given fixed energy problem (1.3) as in [Eke3] instead, we consider the following fixed period problem:

$$\begin{cases} \dot{z}(t) &= JH'_\alpha(z(t)), & \forall t \in \mathbf{R}, \\ z(1) &= z(0). \end{cases} \quad (2.1)$$

Define

$$E = \left\{ u \in L^{(\alpha-1)/\alpha}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^{2n}) \mid \int_0^1 u(t) dt = 0 \right\}. \quad (2.2)$$

The corresponding Clarke-Ekeland dual action function $f : E \rightarrow \mathbf{R}$ is defined by

$$f(u) = \int_0^1 \left(\frac{1}{2}(Ju, Mu) + H_\alpha^*(-Ju) \right) dt, \quad (2.3)$$

where Mu is defined by $\frac{d}{dt}Mu(t) = u(t)$ and $\int_0^1 Mu(t)dt = 0$, and the usual dual function H_α^* of H_α is defined by

$$H_\alpha^*(x) = \sup_{y \in \mathbf{R}^{2n}} ((x, y) - H_\alpha(y)), \quad (2.4)$$

then $f \in C^2(E, \mathbf{R})$. Suppose z is a solution of (2.1), then $u = \dot{z}$ is a critical point of f . Conversely, suppose $u \in E \setminus \{0\}$ is a critical point of f , then there exists a unique $\xi_u \in \mathbf{R}^{2n}$ such that

$z_u(t) = Mu(t) + \xi_u$ is a solution of (2.1). In particular, solutions of (2.1) are in one to one correspondence with critical points of f .

Following §V.3 of [Eke3], we denote by "ind" the Fadell-Rabinowitz S^1 -action cohomology index theory (please refer [FaR]) for S^1 -invariant subsets of E defined in [Eke3]. For $[f]_c \equiv \{u \in E | f(u) \leq c\}$, the following critical values of f are defined

$$c_k = \inf\{c < 0 | \text{ind}([f]_c) \geq k\}, \quad \forall k \in \mathbf{N} \quad (2.5)$$

Ekeland and Hofer proved the following theorem which is a basis of further study [EkH], [Eke3], the theorem with the following form is from [LZh].

Theorem 2.1.

$$\begin{aligned} -\infty < c_1 &= \inf_{u \in E} f(u) \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots < 0 \\ c_k &\rightarrow 0 \text{ as } k \rightarrow +\infty, \\ \#\mathcal{T}(\Sigma) &= +\infty \text{ if } c_k = c_{k+1} \text{ for some } k \in \mathbf{N}. \end{aligned}$$

For any given $k \in \mathbf{N}$, there exists $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ and $m \in \mathbf{N}$ such that for

$$u_m^x(t) = (m\tau)^{(\alpha-1)/(2-\alpha)} \dot{x}(m\tau t), \quad 0 \leq t \leq 1, \quad (2.6)$$

there hold

$$f'(u_m^x) = 0, \quad f(u_m^x) = c_k, \quad (2.7)$$

$$i(x, m) \leq 2k - 2 + n \leq i(x, m) + \nu(x, m) - 1, \quad (2.8)$$

where $i(x, m)$ is the Maslov-type index of closed characteristics x with m -th iteration, for the readers whom are not familiar with the Maslov-type index, please refer Section 4 for the definitions and basic notations.

Definition 2.2.(cf.[LZh]) For any $\Sigma \in \mathcal{H}(2n)$ and $\alpha \in (1, 2)$, $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ is (m, k) -variationally visible, if there exist some m and $k \in \mathbf{N}$ such that (2.7), (2.8) hold for u_m^x defined by (2.6). We call $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ infinite variationally visible, if there exist infinitely many (m, k) such that (τ, x) is (m, k) -variationally visible. We denote by $\mathcal{V}_\infty(\Sigma, \alpha)$ the subset of $\mathcal{J}(\Sigma, \alpha)$ in which a representative $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ of each $[(\tau, x)]$ is infinitely variationally visible.

Theorem 2.3. (cf.[LZh]) There exists an integer $K \geq 0$ and an injection map

$$p : \mathbf{N} + K \longrightarrow \mathcal{V}_\infty(\Sigma, \alpha) \times \mathbf{N}$$

such that

(i) For any $k \in \mathbf{N} + K$, $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ and $m \in \mathbf{N}$ satisfying $p(k) = ([(\tau, x)], m)$, (2.7) and (2.8) hold, and

(ii) For any $k_j \in \mathbf{N} + K$, $k_1 < k_2$, $(\tau_j, x_j) \in \mathcal{J}(\Sigma, \alpha)$ satisfying $p(k_j) = ([(\tau_j, x_j)], m_j)$ with $j = 1, 2$,

$$\hat{i}(x_1, m_1) < \hat{i}(x_2, m_2). \quad (2.9)$$

Remark 2.4 Since $\alpha \in (1, 2)$, Let $\gamma \in \mathcal{P}_\tau(2n)$ be the fundamental matrix of (x, τ) , $M = \gamma(\tau)$, then there exist $P \in \text{Sp}(2n)$ and $Q \in \text{Sp}(2n - 2)$ such that $M = P^{-1}(N_1(1, 1) \diamond Q)P$. Since H_α is convex, then $i_1(\gamma) \geq n$, and consequently the mean Maslov-type index $\hat{i}(\gamma) > 2$ for $n \geq 2$.

A key Theorem of [LZh] is the following index jump theorem.

Theorem 2.5.(cf. P.350 of [LZh]) Let $\gamma_k \in \mathcal{P}_{\tau_k}(2n)$ for $k = 1, \dots, q$ be a finite collection of symplectic paths. Let $M_k = \gamma_k(\tau_k)$. Suppose that there exists $P_k \in \text{Sp}(2n)$ and $Q_k \in \text{Sp}(2n - 2)$ such that $M_k = P_k^{-1}(N_1(1, 1) \diamond Q_k)P_k$ and $\hat{i}(\gamma_k, 1) > 0$, for all $k = 1, \dots, q$. Then there exist infinitely many $(N, m_1, \dots, m_q) \in \mathbf{N}^{q+1}$ such that

$$I(k, m_k) = N + \Delta_k, \quad (2.10)$$

where

$$\begin{aligned} I(k, m_k) &= m_k(i(\gamma_k, 1) + S_{M_k}^+(1) - C(M_k)) \\ &+ \sum_{\theta \in (0, 2\pi)} E\left(\frac{m_k \theta}{\pi}\right) S_{M_k}^-(e^{\sqrt{-1}\theta}), \end{aligned} \quad (2.11)$$

$$\Delta_k = \sum_{0 < \{m_k \frac{\theta}{\pi}\} < \delta} S_{M_k}^-(e^{\sqrt{-1}\theta}) \quad (2.12)$$

for every $k = 1, \dots, q$. Moreover we have

$$\min \left\{ \left\{ \frac{m_k \theta}{\pi} \right\}, 1 - \left\{ \frac{m_k \theta}{\pi} \right\} \right\} < \delta, \quad (2.13)$$

$$m_k \frac{\theta}{\pi} \in \mathbf{Z}, \quad \text{if } \frac{\theta}{\pi} \in \mathbf{Q}, \quad (2.14)$$

where $e^{\sqrt{-1}\theta} \in \sigma(M_k)$, $\frac{\theta}{\pi} \in (0, 2)$ and δ can be chosen as small as we want (cf. (4.43) of [LZh]).

More precisely, by (4.10), (4.40), and (4.41) in [LZh], we have

$$m_k = \left(\left[\frac{N}{M\hat{i}(\gamma_k, 1)} \right] + \chi_k \right) M, \quad 1 \leq k \leq q, \quad (2.15)$$

where $\chi_k = 0$ or 1 for $1 \leq k \leq q$. Furthermore, given $M_0 \in \mathbf{N}$, by the proof of Theorem 4.1 of [LZh], we may further require $M_0|N$ (since the closure of the set $\{\{Nv\} : N \in \mathbf{N}, M_0|N\}$) is still a closed additive subgroup of T^h for some $h \in \mathbf{N}$, where we use the notations as (4.21)-(4.22) in [LZh]. Then we can use the step 2 in Theorem 4.1 of [LZh] to get N .

In fact, by (4.40)-(4.41) of [LZh], let $\mu_i = \sum_{\theta \in (0, 2\pi)} S_{M_i}^-(e^{\sqrt{-1}\theta})$ for $1 \leq i \leq q$ and $\alpha_{i,j} = \frac{\theta_j}{\pi}$ where $e^{\sqrt{-1}\theta_j} \in \sigma(M_i)$ for $1 \leq j \leq \mu_i$ and $1 \leq i \leq q$. As in (4.21) of [LZh], let $h = q + \sum_{1 \leq i \leq q} \mu_i$ and

$$v = \left(\frac{1}{M\hat{i}(\gamma_1, 1)}, \dots, \frac{1}{M\hat{i}(\gamma_q, 1)}, \frac{\alpha_{1,1}}{\hat{i}_1(\gamma_1, 1)}, \frac{\alpha_{1,2}}{\hat{i}_1(\gamma_1, 1)}, \dots, \frac{\alpha_{1,\mu_1}}{\hat{i}_1(\gamma_1, 1)}, \frac{\alpha_{2,1}}{\hat{i}_1(\gamma_2, 1)}, \dots, \frac{\alpha_{q,\mu_q}}{\hat{i}_1(\gamma_q, 1)} \right). \quad (2.16)$$

Then by (4.22) of [LZh], the above theorem is equivalent to find a vertex

$$\chi = (\chi_1, \dots, \chi_q, \chi_{1,1}, \chi_{1,2}, \dots, \chi_{1,\mu_1}, \chi_{2,1}, \dots, \chi_{q,\mu_q}) \quad (2.17)$$

of the cube $[0, 1]^h$ and infinitely many integers $N \in \mathbf{N}$ such that

$$|\{Nv\} - \chi| < \varepsilon \quad (2.18)$$

for any given ε small enough.

Theorem 2.6. (cf. Theorem 4.2 of [LZh]) Let H be the closure of $\{\{mv\} | m \in \mathbf{N}\}$ in $T^h = (\mathbf{R}/Z)^h$ and $V = T_0\pi^{-1}H$ be the tangent space of $\pi^{-1}H$ at the origin in \mathbf{R}^h , where $\pi : \mathbf{R}^h \rightarrow T^h$ is the projection map. Define

$$A(v) = V \setminus \cup_{v_k \in R \setminus Q} \{x = (x_1, \dots, x_h) \in V | x_k = 0\}. \quad (2.19)$$

Define $\psi(x) = 0$ when $x \geq 0$ and $\psi(x) = 1$ when $x < 0$. Then for any $a = (a_1, \dots, a_h) \in A(v)$, the vector

$$\chi(a) = (\psi(a_1), \dots, \psi(a_h)) \quad (2.20)$$

makes (2.18) hold for infinitely many $N \in \mathbf{N}$.

Please note that when we choose $a \in V$ small enough, then $a + \chi(a) \in [0, 1]^h$, this implies $(V + \chi(a)) \cap [0, 1]^h \neq \emptyset$, and so we can require $N \in \mathbf{N}$ in (2.18) satisfied $\{Nv\} - \chi(a) \in V$.

Theorem 2.7.(cf. Theorem 4.2 of [LZh]) We have the following properties for $A(v)$:

- (i) When $v \in \mathbf{R}^h \setminus \mathbf{Q}^h$, then $\dim V \geq 1$, $0 \notin A(v) \subset V$, $A(v) = -A(v)$ and $A(v)$ is open in V .
- (ii) When $\dim V = 1$, then $A(v) = V \setminus \{0\}$.
- (iii) When $\dim V \geq 2$, $A(v)$ is obtained from V by deleting all the coordinate hyperplanes with dimension strictly smaller than $\dim V$ from V .

Remark 2.8. In our choice of (N, m_1, \dots, m_q) in the proof of Theorem 2.5, we can choose M_0 good enough such that $N \in \mathbf{N}$ further satisfies

$$\frac{N}{M\hat{i}(\gamma_k, 1)} \in \mathbf{Z}, \text{ for } \forall \hat{i}(\gamma_k, 1) \in \mathbf{Q}, \quad k \in \{1, \dots, q\}. \quad (2.21)$$

Furthermore from (2.18), we get $\hat{i}(\gamma_k, 1) \in \mathbf{Q}$ implies $\chi_k(a) = \psi(a_k) = 0$.

From the theorems above, we get a useful lemma below, this lemma is very important in our proof of the main Theorem 2.

Lemma 2.9. Let $v = (v_1, v_2, \dots, v_h)$ given by (2.16). If $v_i, v_j \in \mathbf{R} \setminus \mathbf{Q}$ and $\frac{v_j}{v_i} = \frac{p}{q} \in Q^+(i < j)$, then for $\forall x \in V$, we have $\frac{x_j}{x_i} = \frac{p}{q}$ and for $\forall a \in A(v)$, we have $\chi_i(a) = \chi_j(a)$.

Proof. Since $H = \overline{\{\{mv\} | m \in \mathbf{N}\}}$, if $v_i, v_j \in \mathbf{R} \setminus \mathbf{Q}$ and $\frac{v_j}{v_i} = \frac{p}{q} \in Q^+(i < j)$, that means v_i, v_j are rational dependent, then we restrict H to the coordinate hyperplane

$$D = \{(0, \dots, 0, x_i, 0, \dots, 0, x_j, 0, \dots, 0) | x_i, x_j \in R\} \subset R^h,$$

we have the dynamical picture of $H \cap D$ below:

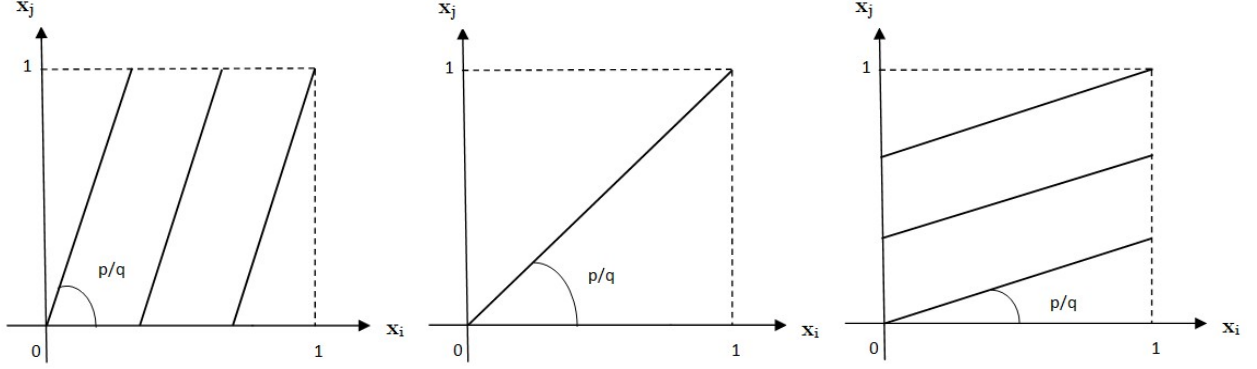


Figure 2.1: If $\frac{v_j}{v_i} = \frac{p}{q} > 1$

Figure 2.2: If $\frac{v_j}{v_i} = \frac{p}{q} = 1$

Figure 2.3: If $\frac{v_j}{v_i} = \frac{p}{q} < 1$

hence for any $x \in V = T_0\pi^{-1}H, x_i \neq 0$, we have $\frac{x_j}{x_i} = \frac{p}{q} \in \mathbf{Q}^+$. In particular, if $a \in A(v)$, then $a_i > 0, a_j > 0$ or $a_i < 0, a_j < 0$, hence $\chi_i(a) = \chi_j(a) = 0$ or $\chi_i(a) = \chi_j(a) = 1$, this completes the proof. \square

Furthermore, as Definition 1.1 of [LZh], we define

Definition 2.10. For $\alpha \in (1, 2)$, we define a map $\varrho_n(\Sigma) : \mathcal{H}(2n) \rightarrow \mathbf{N} \cup \{+\infty\}$

$$\varrho_n(\Sigma) = \begin{cases} +\infty, & \text{if } \#\mathcal{V}(\Sigma, \alpha) = +\infty, \\ \min \left\{ \left\lceil \frac{i(x,1)+2S^+(x)-\nu(x,1)+n}{2} \right\rceil \mid (\tau, x) \in \mathcal{V}(\Sigma, \alpha) \right\}, & \text{if } \#\mathcal{V}(\Sigma, \alpha) < +\infty. \end{cases} \quad (2.22)$$

As in [LZh], we denote the elements of $\mathcal{V}_\infty(\Sigma, \alpha)$ by

$$\mathcal{V}_\infty(\Sigma, \alpha) = \{[(\tau_j, x_j)] \mid j = 1, \dots, q\},$$

where $(\tau_j, x_j) \in \mathcal{J}(\Sigma, \alpha)$ for $j = 1, \dots, q$.

Theorem 2.11. For given $a \in A(v)$, we define $\chi \equiv \chi(a) = (\psi(a_1), \dots, \psi(a_h))$ by (2.20). Let $(N, m_1, \dots, m_q) \in \mathbf{N}^{q+1}$ be given in Remark 2.8. Then by the proof Theorem 5.1 in [LZh], for each $s = 1, \dots, \varrho_n(\Sigma)$, there exists a unique $j(s) \in \{1, \dots, q\}$ and an injection map $p : N + K \rightarrow \mathcal{V}_\infty(\Sigma, \alpha) \times \mathbf{N}$ such that $p(N - s + 1) = ([(\tau_{j(s)}, x_{j(s)})], 2m_{j(s)})$ and

$$i(x_{j(s)}, 2m_{j(s)}) \leq 2N - 2s + n \leq i(x_{j(s)}, 2m_{j(s)}) + \nu(x_{j(s)}, 2m_{j(s)}) - 1, \quad (2.23)$$

Then for any $s_1, s_2 \in \{1, \dots, \varrho_n(\Sigma)\}$ with $s_1 < s_2$, we have:

$$\left(\left\lceil \frac{N}{MD_{j(s_2)}} \right\rceil + \chi_{j(s_2)}(a) \right) MD_{j(s_2)} < \left(\left\lceil \frac{N}{MD_{j(s_1)}} \right\rceil + \chi_{j(s_1)}(a) \right) MD_{j(s_1)} \quad (2.24)$$

and

$$i(x_{j(s)}, 2m_{j(s)}) = 2(N + \Delta_{j(s)}) - (S_{M_{j(s)}}^+(1) + C(M_{j(s)})), \quad (2.25)$$

$$2s \geq n + S_{M_{j(s)}}^+(1) + C(M_{j(s)}) - 2\Delta_{j(s)} - \nu(x_{j(s)}, 2m_{j(s)}) + 1, \quad (2.26)$$

$$2s \leq n + S_{M_{j(s)}}^+(1) + C(M_{j(s)}) - 2\Delta_{j(s)}, \quad (2.27)$$

where $D_{j(s)} = \hat{i}(x_{j(s)}, 1)$.

Proof. Since $s_1 < s_2$, we have $N - s_1 + 1 > N - s_2 + 1$, then from Theorem 2.3, we have

$$\hat{i}(x_{j(s_2)}, 2m_{j(s_2)}) < \hat{i}(x_{j(s_1)}, 2m_{j(s_1)}),$$

the property of the mean index $\hat{i}(x, m) = m\hat{i}(x, 1)$ implies that

$$2m_{j(s_2)}\hat{i}(x_{j(s_2)}, 1) < 2m_{j(s_1)}\hat{i}(x_{j(s_1)}, 1). \quad (2.28)$$

From the definition of $m_{j(s)} = ([\frac{N}{M\hat{i}(\gamma_{j(s)}, 1)}] + \chi_{j(s)}(a))M$ and $D_{j(s)} = \hat{i}(x_{j(s)}, 1)$ we get (2.24). In order to prove formula (2.25), we need some identities (2.10), (2.11) and (4.27) below

$$I(j(s), m_{j(s)}) = N + \Delta_{j(s)},$$

where

$$\begin{aligned} I(j(s), m_{j(s)}) &= m_{j(s)}(i(\gamma_{j(s)}, 1) + S_{M_{j(s)}}^+(1) - C(M_{j(s)})) \\ &+ \sum_{\theta \in (0, 2\pi)} E\left(\frac{m_{j(s)}\theta}{\pi}\right) S_{M_{j(s)}}^-(e^{\sqrt{-1}\theta}), \end{aligned}$$

$$\begin{aligned} i(\gamma_{j(s)}, m_{j(s)}) &= m_{j(s)}(i(\gamma_{j(s)}, 1) + S_{M_{j(s)}}^+(1) - C(M_{j(s)})) \\ &+ 2 \sum_{\theta \in (0, 2\pi)} E\left(\frac{m_{j(s)}\theta}{2\pi}\right) S_{M_{j(s)}}^-(e^{\sqrt{-1}\theta}) - (S_{M_{j(s)}}^+(1) + C(M_{j(s)})). \end{aligned}$$

where $C(M_{j(s)}) = \sum_{0 < \theta < 2\pi} S_{M_{j(s)}}^-(e^{\sqrt{-1}\theta})$.

Simple calculations show that

$$\begin{aligned} i(\gamma_{j(s)}, 2m_{j(s)}) &= 2I(j(s), m_{j(s)}) - (S_{M_{j(s)}}^+(1) + C(M_{j(s)})) \\ &= 2(N + \Delta_{j(s)}) - (S_{M_{j(s)}}^+(1) + C(M_{j(s)})). \end{aligned}$$

hence we get formula (2.25). On the other hand, it's easy to show that formulas (2.23), (2.25) imply (2.26), (2.27). \square

Corollary 2.12. Further properties of inequalities (2.24):

i) If $D_{j(s_2)} \in \mathbf{Q}$, then $\chi_{j(s_2)}(a) = 0$ and

$$N = \left(\left\lfloor \frac{N}{MD_{j(s_2)}} \right\rfloor + \chi_{j(s_2)}(a) \right) MD_{j(s_2)} < \left(\left\lfloor \frac{N}{MD_{j(s_1)}} \right\rfloor + \chi_{j(s_1)}(a) \right) MD_{j(s_1)}$$

with $D_{j(s_1)} \in \mathbf{R} \setminus \mathbf{Q}$, $\chi_{j(s_1)}(a) = 1$.

ii) If $\chi_{j(s_2)}(a) = 1$, then $D_{j(s_2)} \in \mathbf{R} \setminus \mathbf{Q}$ and

$$N < \left(\left\lfloor \frac{N}{MD_{j(s_2)}} \right\rfloor + \chi_{j(s_2)}(a) \right) MD_{j(s_2)} < \left(\left\lfloor \frac{N}{MD_{j(s_1)}} \right\rfloor + \chi_{j(s_1)}(a) \right) MD_{j(s_1)}$$

with $D_{j(s_1)} \in \mathbf{R} \setminus \mathbf{Q}$, $\chi_{j(s_1)}(a) = 1$.

iii)

$$\chi_{j(s_2)}(a) \leq \chi_{j(s_1)}(a). \quad (2.29)$$

Proof. From Remark 2.8, we know that $D_{j(s_2)} \in \mathbf{Q}$ implies $\frac{N}{MD_{j(s_2)}} \in \mathbf{Z}$, then (2.18) implies that $\chi_{j(s_2)}(a) = 0$, hence the formula $(\lfloor \frac{N}{MD_{j(s_2)}} \rfloor + \chi_{j(s_2)}(a))MD_{j(s_2)} = N$. For this case, it's easy to check that inequality (2.24) holds if and only if $D_{j(s_1)} \in \mathbf{R} \setminus \mathbf{Q}$, $\chi_{j(s_1)}(a) = 1$. This completes the proof of i). From i) we know if $D_{j(s_2)} \in \mathbf{Q}$ then $\chi_{j(s_2)}(a) = 0$, so $\chi_{j(s_2)}(a) = 1$ implies that $D_{j(s_2)} \in \mathbf{R} \setminus \mathbf{Q}$, easy computation shows that

$$N = \left(\frac{N}{MD_{j(s_2)}} \right) MD_{j(s_2)} < \left(\left\lfloor \frac{N}{MD_{j(s_2)}} \right\rfloor + \chi_{j(s_2)}(a) \right) MD_{j(s_2)}. \quad (2.30)$$

This combine with inequality (2.24), we get ii). To prove iii), please note that if $\chi_{j(s_2)}(a) = 0$, then (2.29) is obviously right, the case $\chi_{j(s_2)}(a) = 1$ is from ii).

Remark 2.13 It is proved in [LZh] that for $s = 1, \dots, \varrho_n(\Sigma)$, $p(N - s + 1)$ are geometric different, thus there are at least $\varrho_n(\Sigma)$ closed characteristics, and moreover at least $\varrho_n(\Sigma) - 1$ among them have irrational mean index and $p(N)$ is elliptic.

3 Proofs of the Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2 based on the index iteration theory developed by Y. Long and his coworkers. Some notations for the Maslov-type index can be found in Section 4. The

basic normal form $R(\theta_j)$ ($N_2(\omega_j, u_j)$; $N_2(\lambda_j, \nu_j)$) given in Theorem 4.7 is called rational normal form, if $\frac{\theta_j}{\pi} \in \mathbf{Q}$ ($\frac{\alpha_j}{\pi} \in \mathbf{Q}$; $\frac{\beta_j}{\pi} \in \mathbf{Q}$).

Lemma 3.1. For any fix $a \in A(v)$, let inject map $p(N - s + 1) = ([(\tau_{j(s)}, x_{j(s)}], 2m_{j(s)}), s \in \{1, \dots, \varrho_n(\Sigma)\}$ given in Theorem 2.11, if $x_{j(2)}$ is not an elliptic closed characteristic, then $\chi_{j(2)}(a) = 0$ implies that $\hat{i}(x_{j(2)}, 1) \in \mathbf{Q}$.

Proof. From Theorem 4.7, we have the symplectic decomposition

$$\begin{aligned} \gamma_{j(2)} &\simeq N_1(1, 1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1, -1)^{\diamond p_+} \diamond N_1(-1, 1)^{\diamond q_-} \diamond -I_{2q_0} \diamond N_1(-1, -1)^{\diamond q_+} \\ &\diamond R(\theta_1) \diamond \dots \diamond R(\theta_r) \diamond N_2(\omega_1, u_1) \diamond \dots \diamond N_2(\omega_{r_*}, u_{r_*}) \\ &\diamond N_2(\lambda_1, \nu_1) \diamond \dots \diamond N_2(\lambda_{r_0}, \nu_{r_0}) \diamond M_k, \end{aligned} \quad (3.1)$$

for this decomposition, the number of the rational normal form in $\{R(\theta_1), \dots, R(\theta_r)\}$ is denoted by \tilde{r} . Similarly, for set $\{N_2(\omega_1, u_1), \dots, N_2(\omega_{r_*}, u_{r_*})\}$ and $\{N_2(\lambda_1, \nu_1), \dots, N_2(\lambda_{r_0}, \nu_{r_0})\}$, the number of rational normal form is denoted by \tilde{r}_* and \tilde{r}_0 respectively, then from (4.31) we have a further estimation of the variable $\nu(\gamma_{j(2)}, 2m_{j(2)})$ below

$$\begin{aligned} \nu(\gamma_{j(2)}, 2m_{j(2)}) &= \nu(\gamma_{j(2)}, 1) + q_- + 2q_0 + q_+ + 2(r + r_* + r_0) \\ &\quad - 2(r - \tilde{r} + r_* - \tilde{r}_* + r_0 - \tilde{r}_0) \\ &= p_- + 2p_0 + p_+ + q_- + 2q_0 + q_+ + 2(\tilde{r} + \tilde{r}_* + \tilde{r}_0). \end{aligned} \quad (3.2)$$

Now we proof the lemma by contradiction. Assume that $\chi_{j(2)}(a) = 0$ and $\hat{i}(x_{j(2)}, 1) \in \mathbf{R} \setminus \mathbf{Q}$, then (4.32) implies that at least one of $\frac{\theta_1}{\pi}, \frac{\theta_2}{\pi}, \dots, \frac{\theta_r}{\pi}$ is irrational number, hence $r - \tilde{r} \geq 1$ and

$$\begin{aligned} \{m_{j(2)} D_{j(2)}\} &= \left\{ m_{j(2)} \left(i(x_{j(2)}, 1) + p_- + p_0 - r + \sum_{j=1}^r \frac{\theta_j}{\pi} \right) \right\} \\ &= \left\{ m_{j(2)} \sum_{\frac{\theta_j}{\pi} \in \mathbf{R} \setminus \mathbf{Q}} \frac{\theta_j}{\pi} \right\} \\ &\leq \sum_{\frac{\theta_j}{\pi} \in \mathbf{R} \setminus \mathbf{Q}} \left\{ m_{j(2)} \frac{\theta_j}{\pi} \right\}, \end{aligned} \quad (3.3)$$

where $m_{j(2)} = ([\frac{N}{MD_{j(2)}}] + \chi_{j(2)}(a))M = [\frac{N}{MD_{j(2)}}]M$. The second equality comes from (2.14) in Theorem 2.5.

On the other hand,

$$\begin{aligned}\{m_{j(2)}D_{j(2)}\} &= \left\{ \left\lfloor \frac{N}{MD_{j(2)}} \right\rfloor MD_{j(2)} \right\} \\ &= \left\{ N - \left\{ \frac{N}{MD_{j(2)}} \right\} MD_{j(2)} \right\},\end{aligned}\tag{3.4}$$

and from (2.16) and (2.18), we get that $\left\{ \frac{N}{MD_{j(2)}} \right\} = \left| \left\{ \frac{N}{MD_{j(2)}} \right\} - \chi_{j(2)}(a) \right| < \varepsilon$ ($\chi_{j(2)}(a) = 0$) for any given ε small enough, let $\varepsilon < \frac{1-\delta}{MD_{j(2)}}$, , where δ is given in Theorem 2.5, hence $\{m_{j(2)}D_{j(2)}\} > \delta$, this combines with (3.3), we have

$$\delta < \sum_{\frac{\theta_j}{\pi} \in \mathbf{R} \setminus \mathbf{Q}} \left\{ m_{j(2)} \frac{\theta_j}{\pi} \right\},\tag{3.5}$$

hence at least one of the elements in $\{\frac{\theta_j}{\pi} | \frac{\theta_j}{\pi} \in \mathbf{R} \setminus \mathbf{Q}, j = 1, \dots, r\}$ satisfies $\{m_{j(2)} \frac{\theta_j}{\pi}\} \notin (0, \delta)$. We have the estimation of the variable $\Delta_{j(2)}$ below

$$\begin{aligned}\Delta_{j(2)} &= \sum_{0 < \{m_{j(2)} \frac{\theta}{\pi}\} < \delta} S_{M_{j(2)}}^-(e^{\sqrt{-1}\theta}) \\ &\leq \sum_{\frac{\theta}{\pi} \in \mathbf{R} \setminus \mathbf{Q}} S_{M_{j(2)}}^-(e^{\sqrt{-1}\theta}) - 1 \\ &= r - \tilde{r} - 1 + 2(r_* - \tilde{r}_*),\end{aligned}\tag{3.6}$$

where the last equality comes from the calculation of the splitting number of the basic normal given in Definition 4.2. In order to prove the Lemma, we rewrite the useful inequality (2.26) and equalities (4.29), (4.33), (4.34), (3.2) below

$$\begin{aligned}2s &\geq n + S_{M_{j(s)}}^+(1) + C(M_{j(s)}) - 2\Delta_{j(s)} - \nu(x_{j(s)}, 2m_{j(s)}) + 1, \quad (s = 2) \\ n &= p_- + p_0 + p_+ + q_- + q_0 + q_+ + r + 2r_* + 2r_0 + k \\ S_{M_{j(2)}}^+(1) &= p_- + p_0 \\ C(M_{j(2)}) &= \sum_{0 < \theta < 2\pi} S_{M_{j(2)}}^-(e^{\sqrt{-1}\theta}) = q_0 + q_+ + r + 2r_* \\ \nu(\gamma_{j(2)}, 2m_{j(2)}) &= p_- + 2p_0 + p_+ + q_- + 2q_0 + q_+ + 2(\tilde{r} + \tilde{r}_* + \tilde{r}_0)\end{aligned}$$

This combine with inequality (3.6), easy computation shows that for $s = 2$, we have

$$4 \geq p_- + q_+ + 2(r_0 - \tilde{r}_0) + 2\tilde{r}_* + k + 3\tag{3.7}$$

From Remark 2.4, we always have $p_- \geq 1$. On the other hand, from the condition of the lemma, we know $x_{j(2)}$ is not an elliptic closed characteristic, this implies that $k \geq 1$, hence we get $4 \geq 5$.

This contradiction completes the proof. \square

Corollary 3.2. For any fix $a \in A(v)$, let inject map $p(N - s + 1) = ([(\tau_{j(s)}, x_{j(s)}), 2m_{j(s)}), s \in \{1, \dots, \varrho_n(\Sigma)\}$ given in Theorem 2.11. If $x_{j(2)}$ is not elliptic, then $\chi_{j(1)}(a) = 1$ and $\hat{i}(x_{j(1)}, 1) \in \mathbf{R} \setminus \mathbf{Q}$.

Proof. If $\chi_{j(2)}(a) = 1$, from ii) of Corollary 2.12 we have $\chi_{j(1)}(a) = 1$ and $\hat{i}(x_{j(1)}, 1) \in \mathbf{R} \setminus \mathbf{Q}$. If $\chi_{j(2)}(a) = 0$, then from Lemma 3.1, we have $\hat{i}(x_{j(2)}, 1) \in \mathbf{Q}$. This combines with i) of Corollary 2.12 we get $\hat{i}(x_{j(1)}, 1) \in \mathbf{R} \setminus \mathbf{Q}$ and $\chi_{j(1)}(a) = 1$. \square

Now we start to proof Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. For any fix $a \in A(v)$, let inject map $p(N - s + 1) = ([(\tau_{j(s)}, x_{j(s)}), 2m_{j(s)}), s \in \{1, \dots, \varrho_n(\Sigma)\}$, given in Theorem 2.11. From Remark 2.13, we know $x_{j(1)}$ is elliptic. If $x_{j(2)}$ is also elliptic, then the proof is complete. If $x_{j(2)}$ is not an elliptic closed characteristic, from Corollary 3.2, we get $\chi_{j(1)}(a) = 1$ and $\hat{i}(x_{j(1)}, 1) \in \mathbf{R} \setminus \mathbf{Q}$. Now from Theorem 2.7, we can choose $-a \in A(v)$, then Theorem 2.11 says that for $-a \in A(v)$, we still have $(\tilde{N}, \tilde{m}_1, \dots, \tilde{m}_q)$, $\tilde{j}(s)$ and inject map $p(\tilde{N} - s + 1) = ([(\tau_{\tilde{j}(s)}, x_{\tilde{j}(s)}), 2\tilde{m}_{\tilde{j}(s)}), s \in \{1, \dots, \varrho_n(\Sigma)\}$. If $\tilde{j}(1) \neq j(1)$, then $x_{j(1)}, x_{\tilde{j}(1)}$ are two different elliptic closed characteristics, then we complete the proof. If $\tilde{j}(1) = j(1)$, from the definition of $\chi(a)$, we know that $\chi_{j(1)}(a) = 1$ implies $\chi_{j(1)}(-a) = 0$, hence $\chi_{\tilde{j}(1)}(-a) = \chi_{j(1)}(-a) = 0$, this combine with i), iii) of Corollary 2.12, we have $\chi_{\tilde{j}(2)}(-a) = 0$ and $\hat{i}(x_{\tilde{j}(2)}, 1) \in \mathbf{R} \setminus \mathbf{Q}$, but Lemma 3.1 still holds in the case $-a \in A(v)$, that means if $x_{\tilde{j}(2)}$ is not elliptic, we should have $\chi_{\tilde{j}(2)}(-a) = 0$ implies $\hat{i}(x_{\tilde{j}(2)}, 1) \in \mathbf{Q}$, this contradiction completes the proof. \square

Proof of Theorem 1.2. For the $\varrho_n(\Sigma)$ geometrically distinct closed characteristics in Remark 2.13, for any two closed characteristics $[(\tau, x)], [(\tilde{\tau}, \tilde{x})]$, we know that if one of them has rational mean index, then another must has irrational mean index, hence for this case, the theorem is true. Now we can assume that $\hat{i}(x, 1), \hat{i}(\tilde{x}, 1) \in \mathbf{R} \setminus \mathbf{Q}$. For this case, we get the proof by contradiction.

Assume $\frac{\hat{i}(\tilde{x}, 1)}{\hat{i}(x, 1)} = \frac{p}{q} \in \mathbf{Q}^+$, then from Theorem 2.11, we know there exist $s_1, s_2 \in \{1, 2, \dots, q\}$ such that $x = x_{j(s_1)}$ and $\tilde{x} = x_{j(s_2)}$. Without loss of generality, we can assume that $s_1 < s_2$, then we have (2.24)

$$\left(\left[\frac{N}{MD_{j(s_2)}} \right] + \chi_{j(s_2)}(a) \right) MD_{j(s_2)} < \left(\left[\frac{N}{MD_{j(s_1)}} \right] + \chi_{j(s_1)}(a) \right) MD_{j(s_1)},$$

hence we have

$$\left(\left\{ \frac{N}{MD_{j(s_2)}} \right\} - \chi_{j(s_2)}(a) \right) MD_{j(s_2)} > \left(\left\{ \frac{N}{MD_{j(s_1)}} \right\} - \chi_{j(s_1)}(a) \right) MD_{j(s_1)}, \quad (3.8)$$

where $D_{j(s)} = \hat{i}(x_{j(s)}, 1)$. Let

$$v = \left(\frac{1}{M\hat{i}(\gamma_1, 1)}, \dots, \frac{1}{M\hat{i}(\gamma_q, 1)}, \frac{\alpha_{1,1}}{\hat{i}_1(\gamma_1, 1)}, \frac{\alpha_{1,2}}{\hat{i}_1(\gamma_1, 1)}, \dots, \frac{\alpha_{1,\mu_1}}{\hat{i}_1(\gamma_1, 1)}, \frac{\alpha_{2,1}}{\hat{i}_1(\gamma_2, 1)}, \dots, \frac{\alpha_{q,\mu_q}}{\hat{i}_1(\gamma_q, 1)} \right)$$

given by (2.16), where γ_k is the associated symplectic path of $[(\tau_k, x_k)] \in \mathcal{V}_\infty(\Sigma, \alpha)$, then we have

$$\frac{v_{j(s_2)}}{v_{j(s_1)}} = \frac{\frac{1}{M\hat{i}(\gamma_{j(s_2)}, 1)}}{\frac{1}{M\hat{i}(\gamma_{j(s_1)}, 1)}} = \frac{\hat{i}(x, 1)}{\hat{i}(\tilde{x}, 1)} = \frac{q}{p}. \quad (3.9)$$

On the other hand, from Theorem 2.6, for fixed $a \in V$, we choose $N \in \mathbf{N}$ such that $\{Nv\} - \chi(a)$ small enough, recall that we also have $\{Nv\} - \chi(a) \in V$. From Lemma 2.9, we we get

$$\frac{\{Nv_{j(s_2)}\} - \chi_{j(s_2)}(a)}{\{Nv_{j(s_1)}\} - \chi_{j(s_1)}(a)} = \frac{\left\{ \frac{N}{MD_{j(s_2)}} \right\} - \chi_{j(s_2)}(a)}{\left\{ \frac{N}{MD_{j(s_1)}} \right\} - \chi_{j(s_1)}(a)} = \frac{q}{p}. \quad (3.10)$$

(3.9), (3.10) imply

$$\left(\left\{ \frac{N}{MD_{j(s_1)}} \right\} - \chi_{j(s_1)}(a) \right) MD_{j(s_1)} = \left(\left\{ \frac{N}{MD_{j(s_2)}} \right\} - \chi_{j(s_2)}(a) \right) MD_{j(s_2)} \quad (3.11)$$

This contradiction with (3.8), then the proof is complete. \square

4 Appendix: index iteration theory for closed characteristics

In this section, we briefly recall the index theory for symplectic paths. The index theory is introduced by Conley and Zehnder [CoZ] and developed by Long and others (see [Lon5] for details).

As usual, the symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbf{R}) \mid M^T J M = J\},$$

whose topology is induced from that of \mathbf{R}^{4n^2} . For $\tau > 0$ we are interested in paths in $\mathrm{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0, \tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\},$$

which is equipped with the topology induced from that of $\mathrm{Sp}(2n)$. The following real function was introduced in [Lon4]:

$$D_\omega(M) = (-1)^{n-1} \bar{\omega}^n \det(M - \omega I_{2n}), \quad \forall \omega \in \mathbf{U}, M \in \mathrm{Sp}(2n).$$

Thus for any $\omega \in \mathbf{U}$ the following codimension 1 hypersurface in $\mathrm{Sp}(2n)$ is defined in [Lon4]:

$$\mathrm{Sp}(2n)_\omega^0 = \{M \in \mathrm{Sp}(2n) \mid D_\omega(M) = 0\}.$$

For any $M \in \mathrm{Sp}(2n)_\omega^0$, we define a co-orientation of $\mathrm{Sp}(2n)_\omega^0$ at M by the positive direction $\frac{d}{dt}Me^{t\epsilon J}|_{t=0}$ of the path $Me^{t\epsilon J}$ with $0 \leq t \leq 1$ and $\epsilon > 0$ being sufficiently small. Let

$$\begin{aligned} \mathrm{Sp}(2n)_\omega^* &= \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)_\omega^0, \\ \mathcal{P}_{\tau,\omega}^*(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \mid \gamma(\tau) \in \mathrm{Sp}(2n)_\omega^*\}, \\ \mathcal{P}_{\tau,\omega}^0(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}_{\tau,\omega}^*(2n). \end{aligned}$$

For any two continuous arcs ξ and $\eta : [0, \tau] \rightarrow \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, it is defined as usual:

$$\eta * \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\ \eta(2t - \tau), & \text{if } \tau/2 \leq t \leq \tau. \end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon5], the \diamond -product of M_1 and M_2 is defined by the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k -fold \diamond -product $M \diamond \cdots \diamond M$. Note that the \diamond -product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1 , let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0, \tau]$.

A special path $\xi_n \in \mathcal{P}_\tau(2n)$ is defined by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n} \quad \text{for } 0 \leq t \leq \tau. \quad (4.1)$$

Definition 4.1. (cf. [Lon4], [Lon5]) *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define*

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I_{2n}). \quad (4.2)$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)). \quad (4.3)$$

If $\gamma \in \mathcal{P}_{\tau,\omega}^*(2n)$, define

$$i_\omega(\gamma) = [\mathrm{Sp}(2n)_\omega^0 : \gamma * \xi_n], \quad (4.4)$$

where the right hand side of (4.4) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}_{\tau,\omega}^0(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\{i_\omega(\beta) \mid \beta \in U \cap \mathcal{P}_{\tau,\omega}^*(2n)\}. \quad (4.5)$$

Then

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \dots, 2n\},$$

is called the index function of γ at ω .

For any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbf{N}$, we define its m -th iteration $\gamma^m : [0, m\tau] \rightarrow \mathrm{Sp}(2n)$ by

$$\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for } j\tau \leq t \leq (j+1)\tau, \ j = 0, 1, \dots, m-1. \quad (4.6)$$

We still denote the extended path on $[0, +\infty)$ by γ .

Definition 4.2 (cf. [Lon4], [Lon5]) For any $\gamma \in \mathcal{P}_\tau(2n)$, we define

$$(i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m)), \quad \forall m \in \mathbf{N}. \quad (4.7)$$

The mean index $\hat{i}(\gamma, m)$ per $m\tau$ for $m \in \mathbf{N}$ is defined by

$$\hat{i}(\gamma, m) = \lim_{k \rightarrow +\infty} \frac{i(\gamma, mk)}{k}. \quad (4.8)$$

For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, the splitting numbers $S_M^\pm(\omega)$ of M at ω are defined by

$$S_M^\pm(\omega) = \lim_{\epsilon \rightarrow 0^+} i_{\omega \exp(\pm \sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma), \quad (4.9)$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

For $\Sigma \in \mathcal{H}(2n)$ and $\alpha \in (1, 2)$, let $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$. we define

$$S^+(x) = S_{\gamma_x(\tau)}^+(1), \quad (4.10)$$

$$(i(x, m), \nu(x, m)) = (i(\gamma_x, m), \nu(\gamma_x, m)), \quad (4.11)$$

$$\hat{i}(x, m) = \hat{i}(\gamma_x, m), \quad (4.12)$$

For all $m \in \mathbf{N}$, where γ_x is the associated symplectic path of (τ, x) .

For a given path $\gamma \in \mathcal{P}_\tau(2n)$ we consider to deform it to a new path η in $\mathcal{P}_\tau(2n)$ so that

$$i_1(\gamma^m) = i_1(\eta^m), \quad \nu_1(\gamma^m) = \nu_1(\eta^m), \quad \forall m \in \mathbf{N}, \quad (4.13)$$

and that $(i_1(\eta^m), \nu_1(\eta^m))$ is easy enough to compute. This leads to finding homotopies $\delta : [0, 1] \times [0, \tau] \rightarrow \text{Sp}(2n)$ starting from γ in $\mathcal{P}_\tau(2n)$ and keeping the end points of the homotopy always stay in a certain suitably chosen maximal subset of $\text{Sp}(2n)$ so that (4.13) always holds. In fact, this set was first discovered in [Lon2] as the path connected component $\Omega^0(M)$ containing $M = \gamma(\tau)$ of the set

$$\begin{aligned} \Omega(M) = \{N \in \text{Sp}(2n) \mid \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and} \\ \nu_\lambda(N) = \nu_\lambda(M), \forall \lambda \in \sigma(M) \cap \mathbf{U}\}. \end{aligned} \quad (4.14)$$

Here $\Omega^0(M)$ is called the *homotopy component* of M in $\text{Sp}(2n)$.

In [Lon2]-[Lon5], the following symplectic matrices were introduced as *basic normal forms*:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda = \pm 2, \quad (4.15)$$

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1, b = \pm 1, 0, \quad (4.16)$$

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (4.17)$$

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (4.18)$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ and $b_2 \neq b_3$, $\omega = e^{\sqrt{-1}\theta}$. We call $N_2(\omega, b)$ is nontrivial if $(b_2 - b_3) \sin \theta < 0$ and $N_2(\omega, b)$ is trivial if $(b_2 - b_3) \sin \theta > 0$.

Splitting numbers possess the following properties:

Lemma 4.3. (cf. [Lon2] and Lemma 9.1.5 of [Lon5]) *Splitting numbers $S_M^\pm(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$ appeared in (4.9). For $\omega \in \mathbf{U}$ and $M \in \text{Sp}(2n)$, splitting numbers $S_N^\pm(\omega)$ are constant for all $N \in \Omega^0(M)$.*

Lemma 4.4. (cf. [Lon2], Lemma 9.1.5 and List 9.1.12 of [Lon5]) *For $M \in \text{Sp}(2n)$ and $\omega \in \mathbf{U}$, there hold*

$$S_M^\pm(\omega) = 0, \quad \text{if } \omega \notin \sigma(M). \quad (4.19)$$

$$(S_{N_1(1,a)}^+(1), S_{N_1(1,a)}^-(1)) = \begin{cases} (1, 1), & \text{if } a \geq 0, \\ (0, 0), & \text{if } a < 0. \end{cases} \quad (4.20)$$

$$(S_{N_1(-1,a)}^+(-1), S_{N_1(-1,a)}^-(-1)) = \begin{cases} (1, 1), & \text{if } a \leq 0, \\ (0, 0), & \text{if } a > 0. \end{cases} \quad (4.21)$$

$$(S_{R(\theta)}^+(e^{\sqrt{-1}\theta}), S_{R(\theta)}^-(e^{\sqrt{-1}\theta})) = (0, 1) \text{ if } e^{\sqrt{-1}\theta} \in \mathbf{U} \setminus \mathbf{R}. \quad (4.22)$$

If $e^{\sqrt{-1}\theta} \in \mathbf{U} \setminus \mathbf{R}$ and $N_2(\omega, b)$ is nontrivial, then

$$(S_{N_2(\omega,b)}^+(e^{\sqrt{-1}\theta}), S_{N_2(\omega,b)}^-(e^{\sqrt{-1}\theta})) = (1, 1). \quad (4.23)$$

If $e^{\sqrt{-1}\theta} \in \mathbf{U} \setminus \mathbf{R}$ and $N_2(\omega, b)$ is trivial, then

$$(S_{N_2(\omega,b)}^+(e^{\sqrt{-1}\theta}), S_{N_2(\omega,b)}^-(e^{\sqrt{-1}\theta})) = (0, 0). \quad (4.24)$$

For any $M_i \in \text{Sp}(2n_i)$ with $i = 0$ and 1 , there holds

$$S_{M_0 \diamond M_1}^\pm(\omega) = S_{M_0}^\pm(\omega) + S_{M_1}^\pm(\omega) \text{ and } S_M^\pm(\omega) = S_M^\mp(\bar{\omega}), \quad \forall \omega \in \mathbf{U}. \quad (4.25)$$

where $\bar{\omega}$ is the conjugate of ω . Then we have the following

Theorem 4.5. (cf. [Lon4] and Theorem 1.8.10 of [Lon5]) *For any $M \in \text{Sp}(2n)$, there is a path $f : [0, 1] \rightarrow \Omega^0(M)$ such that $f(0) = M$ and*

$$f(1) = M_1 \diamond \cdots \diamond M_k, \quad (4.26)$$

where each M_i is a basic normal form listed in (4.15)-(4.18) for $1 \leq i \leq k$.

The following is the precise index iteration formulae for symplectic paths, which is due to Y.Long (cf. Chapter 8 [Lon5] or Theorem 2.1, 6.5 and 6.7 of [LZh])

Theorem 4.6. For $n \in \mathbf{N}$, $\tau > 0$, and any path $\gamma \in \mathcal{P}_\tau(2n)$, set $M = \gamma(\tau)$. Extend γ to the whole $[0, +\infty)$. Then for any $m \in \mathbf{N}$,

$$\begin{aligned} i(\gamma, m) &= m(i(\gamma, 1) + S_M^+(1) - C(M)) \\ &\quad + 2 \sum_{\theta \in (0, 2\pi)} E\left(\frac{m\theta}{2\pi}\right) S_M^-(e^{\sqrt{-1}\theta}) - (S_M^+(1) + C(M)). \end{aligned} \quad (4.27)$$

where $C(M) = \sum_{0 < \theta < 2\pi} S_M^-(e^{\sqrt{-1}\theta})$.

Theorem 4.7. Let $\gamma \in \mathcal{P}_\tau(2n)$. Then there exists a path $f \in C([0, 1], \Omega^0(\gamma(\tau)))$ such that $f(0) = \gamma(\tau)$ and

$$\begin{aligned} f(1) &= N_1(1, 1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1, -1)^{\diamond p_+} \diamond N_1(-1, 1)^{\diamond q_-} \diamond -I_{2q_0} \diamond N_1(-1, -1)^{\diamond q_+} \\ &\diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \diamond N_2(\omega_1, u_1) \diamond \cdots \diamond N_2(\omega_{r_*}, u_{r_*}) \\ &\diamond N_2(\lambda_1, v_1) \diamond \cdots \diamond N_2(\lambda_{r_0}, v_{r_0}) \diamond M_k. \end{aligned} \quad (4.28)$$

where $N_2(\omega_j, u_j)$ are non-trivial form with some $\omega_j = e^{\sqrt{-1}\alpha_j}$, $\alpha_j \in (0, \pi) \cup (\pi, 2\pi)$ and $u_j = \begin{pmatrix} u_{j1} & u_{j2} \\ u_{j3} & u_{j4} \end{pmatrix} \in \mathbf{R}^{2 \times 2}$, $N_2(\omega_j, u_j)$ are trivial form with some $\lambda_j = e^{\sqrt{-1}\beta_j}$, $\beta_j \in (0, \pi) \cup (\pi, 2\pi)$ and $v_j = \begin{pmatrix} v_{j1} & v_{j2} \\ v_{j3} & v_{j4} \end{pmatrix} \in \mathbf{R}^{2 \times 2}$, $M_k = D(2)^k$ or $D(-2) \diamond D(2)^{\diamond(k-1)}$; $p_-, p_0, p_+, q_-, q_0, q_+, r, r_*$ and r_0 are non-negative integers; these integer and real number are uniquely determined by $\gamma(\tau)$. It holds that

$$n = p_- + p_0 + p_+ + q_- + q_0 + q_+ + r + 2r_* + 2r_0 + k. \quad (4.29)$$

We also have $i(\gamma, 1)$ is odd if $f(1) = N_1(1, 1), I_2, N_1(-1, 1), -I_2, N_1(-1, -1)$ and $R(\theta)$; $i(\gamma, 1)$ is even if $f(1) = N_1(1, -1)$ and $N_2(\omega, b)$; $i(\gamma, 1)$ can be any integer if $\sigma(f(1)) \cap \mathbf{U} = \emptyset$. Then using the functions defined in (1.7), we have

$$\begin{aligned} i(\gamma, m) &= m(i(\gamma, 1) + p_- + p_0 - r) + 2 \sum_{j=1}^r E\left(\frac{m\theta_j}{2\pi}\right) - r - p_- - p_0 \\ &- \frac{1 + (-1)^m}{2}(q_0 + q_+) + 2\left(\sum_{j=1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) - r_*\right) \end{aligned} \quad (4.30)$$

$$\begin{aligned} \nu(\gamma, m) &= \nu(\gamma, 1) + \frac{1 + (-1)^m}{2}(q_- + 2q_0 + q_+) + 2(r + r_* + r_0) \\ &- 2\left(\sum_{j=1}^r \varphi\left(\frac{m\theta_j}{2\pi}\right) + \sum_{j=1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) + \sum_{j=1}^{r_0} \varphi\left(\frac{m\beta_j}{2\pi}\right)\right) \end{aligned} \quad (4.31)$$

$$\hat{i}(\gamma, 1) = i(\gamma, 1) + p_- + p_0 - r + \sum_{j=1}^r \frac{\theta_j}{\pi} \quad (4.32)$$

$$S_M^+(1) = p_- + p_0 \quad (4.33)$$

$$C(M) = \sum_{0 < \theta < 2\pi} S_M^-(e^{\sqrt{-1}\theta}) = q_0 + q_+ + r + 2r_*. \quad (4.34)$$

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